

Exercise 6A

1 a $y = 2 \cos x$

$$\frac{dy}{dx} = 2 \times (-\sin x) = -2 \sin x$$

b $y = 2 \sin \frac{1}{2}x$

$$\frac{dy}{dx} = 2 \times \frac{1}{2} \cos \frac{1}{2}x = \cos \frac{1}{2}x$$

c $y = \sin 8x$

$$\frac{dy}{dx} = 8 \cos 8x$$

d $y = 6 \sin \frac{2}{3}x$

$$\frac{dy}{dx} = 6 \times \frac{2}{3} \cos \frac{2}{3}x = 4 \cos \frac{2}{3}x$$

2 a $f(x) = 2 \cos x$

$$f'(x) = 2 \times (-\sin x) = -2 \sin x$$

b $f(x) = 6 \cos \frac{5}{6}x$

$$f'(x) = 6 \times \left(-\frac{5}{6} \sin \frac{5}{6}x\right) = -5 \sin \frac{5}{6}x$$

c $f(x) = 4 \cos \frac{1}{2}x$

$$f'(x) = 4 \times \left(-\frac{1}{2} \sin \frac{1}{2}x\right) = -2 \sin \frac{1}{2}x$$

d $f(x) = 3 \cos 2x$

$$f'(x) = 3(-2 \sin 2x) = -6 \sin 2x$$

3 a $y = \sin 2x + \cos 3x$

$$\begin{aligned} \frac{dy}{dx} &= 2 \cos 2x + (-3 \sin 3x) \\ &= 2 \cos 2x - 3 \sin 3x \end{aligned}$$

b $y = 2 \cos 4x - 4 \cos x + 2 \cos 7x$

$$\begin{aligned} \frac{dy}{dx} &= 2 \times (-4 \sin 4x) - 4 \times (-\sin x) \\ &\quad + 2 \times (-7 \sin 7x) \\ &= -8 \sin 4x + 4 \sin x - 14 \sin 7x \end{aligned}$$

c $y = x^2 + 4 \cos 3x$

$$\frac{dy}{dx} = 2x + 4(-3 \sin 3x) = 2x - 12 \sin 3x$$

3 d $y = \frac{1 + 2x \sin 5x}{x} = \frac{1}{x} + 2 \sin 5x$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{x^2} + 2 \times (5 \cos 5x) \\ &= -\frac{1}{x^2} + 10 \cos 5x \end{aligned}$$

4 $y = x - \sin 3x$

$$\frac{dy}{dx} = 1 - 3 \cos 3x$$

At stationary points $\frac{dy}{dx} = 0$

$1 - 3 \cos 3x = 0$

$\cos 3x = \frac{1}{3}$

$3x = 1.23\dots, 5.05\dots \text{ or } 7.51\dots$

$x = 0.410, 1.68 \text{ or } 2.50 \text{ (3 s.f.)}$

$x = 0.410 \Rightarrow y = 0.41 - \sin 1.23 = -0.532$

$x = 1.68 \Rightarrow y = 1.68 - \sin 5.04 = 2.63$

$x = 2.50 \Rightarrow y = 2.50 - \sin 7.50 = 1.56$

Stationary points in the interval $0 \leq x \leq \pi$ are (0.410, -0.532), (1.68, 2.63) and (2.50, 1.56).

5 $y = 2 \sin 4x - 4 \cos 2x$

$$\begin{aligned} \frac{dy}{dx} &= 2 \times 4 \cos 4x - 4 \times (-2 \sin 2x) \\ &= 8 \cos 4x + 8 \sin 2x \end{aligned}$$

When $x = \frac{\pi}{2}$:

$$\begin{aligned} \frac{dy}{dx} &= 8 \cos 2\pi + 8 \sin \pi \\ &= 8 \times 1 + 8 \times 0 = 8 \end{aligned}$$

So the gradient of the curve at the point

where $x = \frac{\pi}{2}$ is 8.

$$6 \quad y = 2 \sin 2x + \cos 2x$$

$$\frac{dy}{dx} = 2 \times 2 \cos 2x + (-2 \sin 2x)$$

$$= 4 \cos 2x - 2 \sin 2x$$

At stationary points $\frac{dy}{dx} = 0$

$$4 \cos 2x - 2 \sin 2x = 0$$

$$4 - 2 \tan 2x = 0$$

$$\tan 2x = 2$$

$$2x = 1.107\dots \text{ or } 4.248\dots$$

$$x = 0.554 \text{ or } 2.12 \text{ (3 s.f.)}$$

When $x = 0.554$:

$$y = 2 \sin(2 \times 0.554) + \cos(2 \times 0.554) = 2.24$$

When $x = 2.12$:

$$y = 2 \sin(2 \times 2.12) + \cos(2 \times 2.12) = -2.24$$

Stationary points in the interval $0 \leq x \leq \pi$ are $(0.554, 2.24)$ and $(2.12, -2.24)$.

$$7 \quad y = \sin 5x + \cos 3x$$

$$\frac{dy}{dx} = 5 \cos 5x - 3 \sin 3x$$

At $(\pi, -1)$, $\frac{dy}{dx} = 5 \cos 5\pi - 3 \sin 3\pi$

$$= 5 \times (-1) - 3 \times 0 = -5$$

Equation of tangent is $y - (-1) = -5(x - \pi)$

or $y = -5x + 5\pi - 1$

$$8 \quad y = 2x^2 - \sin x$$

$$\frac{dy}{dx} = 4x - \cos x$$

When $x = \pi$, $y = 2\pi^2$ and

$$\frac{dy}{dx} = 4\pi - \cos \pi = 4\pi + 1$$

Gradient of normal is $-\frac{1}{4\pi + 1}$

Equation of normal is

$$y - 2\pi^2 = -\frac{1}{4\pi + 1}(x - \pi)$$

Multiplying through by $(4\pi + 1)$ and rearranging gives

$$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$$

$$9 \quad \text{Let } f(x) = \sin x$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\left(\frac{\cos h - 1}{h} \right) \sin x + \left(\frac{\sin h}{h} \right) \cos x \right)$$

Since $\frac{\cos h - 1}{h} \rightarrow 0$ and $\frac{\sin h}{h} \rightarrow 1$,

the expression inside the limit tends to $(0 \times \sin x + 1 \times \cos x)$

So $\lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$.

Challenge

Let $f(x) = \sin(kx)$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sin(kx+kh) - \sin(kx)}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sin kx \cos kh + \cos kx \sin kh - \sin kx}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\left(\frac{\cos kh - 1}{h} \right) \sin kx + \left(\frac{\sin kh}{h} \right) \cos kx \right)$$

As $h \rightarrow 0$, $\left(\frac{\sin kh}{h} \right) \rightarrow k$ and $\left(\frac{\cos kh - 1}{h} \right) \rightarrow 0$,

so the expression inside the limit tends to $0 \times \sin kx + k \times \cos kx = k \cos kx$

Hence the derivative of $\sin(kx)$ is $k \cos(kx)$.